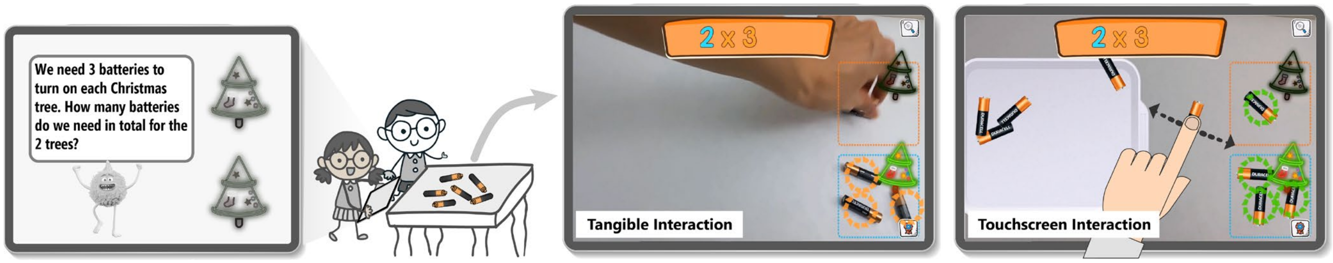
ARMath: Augmenting Everyday Life with Math Learning

Seokbin Kang[1], Ekta Shokeen[2], Virginia L. Byrne[3], Leyla Norooz[2], Elizabeth Bonsignore[2], Caro Williams-Pierce[2], Jon E. Froehlich[4]

Computer Science[1], iSchool[2], Education[3]
University of Maryland, College Park
{sbkang, eshokeen, vbyrne, leylan, ebonsign, carowp}@umd.edu

Allen School of Computer Science and Engineering[4]
University of Washington
jonf@cs.uw.edu



We need 3 batteries to turn on each Christmas tree. How many batteries do we need in total for the 2 trees?

Figure 1. We introduce ARMath, a mobile augmented-reality (AR) system, which recognizes everyday objects and uses life-relevant situations for children to discover and solve math problems. A virtual agent presents a story, such as needing batteries to turn on Christmas trees. Children interactively perform the multiplication problem, 2 (trees) * 3 (batteries), either by directly manipulating physical batteries or moving virtual batteries on the touchscreen. See supplementary video.

ABSTRACT

We introduce ARMath, a mobile Augmented Reality (AR) system that allows children to discover mathematical concepts in familiar, ordinary objects and engage with math problems in meaningful contexts. Leveraging advanced computer vision, ARMath recognizes everyday objects, visualizes their mathematical attributes, and turns them into tangible or virtual manipulatives. Using the manipulatives, children can solve problems that situate math operations or concepts in specific everyday contexts. Informed by four participatory design sessions with teachers and children, we developed five ARMath modules to support basic arithmetic and 2D geometry. We also conducted an exploratory evaluation of ARMath with 27 children (ages 5-8) at a local children's museum. Our findings demonstrate how ARMath engages children in math learning, how failures in AI can be used as learning opportunities, and challenges that children face when using ARMath.

Author Keywords

Augmented Reality; Human-AI Interaction; Learning;

CSS Concepts

Human-centered computing → Mixed / augmented reality

INTRODUCTION

Tangible manipulatives such as blocks and puzzles have long been used in elementary mathematics to promote exploration and understanding of abstract concepts [65,79]. Recent research suggests that using familiar, life-relevant objects engages children in applying math skills and promotes math relevance [53,60]. With advances in computer vision (CV) and augmented reality (AR), we now have an opportunity to explore how to link traditional math learning to everyday experiences. While emerging research in AR-based math learning has focused on immersive visualizations for 3D geometry exploration [45], non-symbolic number training [5], and virtual tutors [68], we explore the integration of everyday objects, virtual storytelling, and AR-based scaffolds.

In this paper, we introduce and evaluate *ARMath*, a mobile AR system for children (K-3) that recognizes everyday objects, turns the objects into math manipulatives, and presents a virtual situation in which children can solve a math problem. ARMath is comprised of four components: (i) a *perception engine* that recognizes objects and their mathematical attributes, (ii) a *problem generator* that presents stories, word problems, and formulas tailored to the objects, (iii) an *interaction engine* that supports interaction with physical or virtual objects for problem solving, and (iv) a *scaffolding engine* that provides audio-visual guidance, procedural feedback, and virtual math tools. With ARMath, children can explore both the mathematical composition of everyday objects—for example, the angles of a book with an AR protractor—as well as use the manipulatives to interactively solve arithmetic problems such as counting physical coins to purchase a virtual ice cream treat.

As initial work, our research questions are exploratory: What are the opportunities of using everyday objects for math learning with AR? What aspects of ARMath seem to engage children in the mathematization experience? What

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are the design implications for AR-based math learning tools? Our research is inspired and informed by prior AR learning systems that demonstrate the potential of turning familiar environments into personally meaningful and engaging learning spaces [46,72,87,95]. We extend the research in three ways. First, to promote relevance of learning, our approach leverages objects existing in everyday life beyond specialized tangible objects [18,74] or locations [17,40]. Second, we target young children (grades K-3) who are less likely to see connections between their daily life and mathematical concepts. [65,71]. Lastly, to inform the design of user interaction, we compare tangible and virtual manipulatives that co-exist in AR.

To create ARMath, we employed an iterative and human-centered design process involving four participatory design sessions (two with teachers, two with children). In the teacher-based sessions we co-designed ARMath learning activities and critiqued existing AR learning tools. For the sessions with children, we examined early user interfaces, which integrated the teachers' ideas, solicited feedback, and cultivated new design ideas, which were integrated into a final ARMath system.

To evaluate ARMath, we conducted five single-session user studies at a local children's museum: 27 children participated (ages 5-8). In our analyses of video recordings, pre- and post-activity questionnaires, and focus groups, we found that children were physically and cognitively engaged with ARMath, actively used scaffolding features, and felt that they had learned mathematical concepts. Interestingly, our findings also highlight how failures in AI can be used as learning opportunities, transforming the child from learner to teacher. However, children also struggled with cognitive gaps between physical and AR worlds, certain AR-assisted interactions (*e.g.,* physically manipulating objects while also viewing the AR tablet screen), and a shortage in conceptual scaffolds.

In summary, our contributions include: first, introducing a real-time mobile AR system for mathematizing everyday experiences; second, enumerating design implications through participatory design studies with teachers and children; and lastly, reporting evaluation results and reflections about the opportunistic use of everyday objects for math learning, tangible *vs.* virtual interactions, and learning with imperfect AI technology.

RELATED WORK

ARMath is informed by research in mathematics education, AR approaches to STEM learning, and hybrid math learning systems.

Mathematizing Life

Recognizing and applying mathematical ideas in everyday life—*i.e.,* mathematizing the world—is critical in math education [47,77,91]. Prior work has shown that the mathematization process can deepen conceptual understanding and promote long-term engagement [65,76]. ARMath supports life-relevant mathematics learning by building on current mathematization practices in formal and informal learning environments.

In formal learning environments, teachers use several material and instructional approaches including: math word problems that illustrate realistic contexts [92], life-relevant references that directly exemplify mathematical concepts [26], and hands-on activities to actively discover math concepts [93]. ARMath builds upon these learning approaches by integrating virtual agents, storytelling, and interactive problem-solving with everyday objects to help motivate and contextualize math learning.

Children's mathematizing experiences also emerge during their play at home [3,90], *e.g.,* counting or sorting toys. In these informal settings, prior work suggests learners benefit from: (i) directing attention to mathematics during real-life activities [82]; (ii) adult intervention to scaffold learning [57]; and (iii) exploration through unstructured manipulation of objects [13]. Using these informal learning attributes, ARMath integrates explicit math tasks (*e.g.,* drawing a shape, counting) and computer-mediated scaffolds that help understand abstract concepts.

ARMath leverages everyday objects as tangible manipulatives to facilitate learning abstract math concepts [89]. Using tangibles poses significant challenges in practice, particularly due to children's difficulties in perceiving and understanding the relationship between the concrete manipulates and the abstract mathematical concepts [61]. Our work explores the potential of AR to aid understanding conceptual relationships between concrete manipulatives and math ideas [12].

In sum, our study explores a mobile AR approach that enables children to mathematize the world around them.

Mobile AR for Math Learning

Our study explores the use of everyday objects for AR-based learning. To provide interactive and contextual learning experiences [37], AR learning systems such as in physics [21,44], chemistry [25,27,83], and electronics [20,36,59], generally employ one of three interaction approaches including: (i) *tangible objects* such as fiducial markers [18,74] or fabricated models [27] that allow for direct manipulation of virtual content; (ii) *user's bodily action* such as hand gestures [48] or whole-body movements [21,69] that can represent dynamic behavior; and (iii) *locations* based on GPS data [17,40] that present location-specific virtual content or learning activities. ARMath extends the tangible approach by exploring the potential of everyday objects specifically for math learning.

While prior work suggests that AR-based math tools support active and social learning via rich information [5,45], little work thus far highlights the role of AR in supporting mathematizing experiences. Prior work mostly focuses on interactive and immersive visualizations, suggesting their benefits of enhancing conceptual understanding of 3D spatial problems [44,45], dimensional analysis [22], or non-numerical magnitude [5]. Only a formative study by Bujak *et al.* [12] suggested the potential of AR to support mathematical discovery in the learner's own environment. Building upon this, ARMath focuses on utilizing physical environments, including physical objects and their mathematical or life-relevant attributes, to blend mathematical ideas and skill into everyday experiences.

Hybrid Mathematics Learning Systems

ARMath also draws inspiration from hybrid math learning systems such as *TICLE* [78] and *BlackBlocks* [2], which combine tangible manipulatives and virtual feedback to support interactive exploration of mathematical ideas. One common drawback, however, is that they require specialized tangible artifacts equipped with sensing capability. For example, *Combinatorix* [80] and *Tangible Tens* [24] require using tangible blocks with visual markers on interactive tabletops. *Representing Equality* [49] uses a balance beam equipped with electronic sensors. This reliance on specialized tangibles limits widespread deployment and affords only a particular type of learning associated with the tangible. ARMath, on the other hand, is mobile and does not require specialized artifacts, leveraging state-of-the-art computer vision techniques to turn everyday objects into math manipulatives anywhere.

Tangible vs. Virtual Interaction

In ARMath, a key design focus was to include two types of user interaction modes—tangible manipulations of physical objects or virtual manipulations on the touchscreen. Despite the importance of this design decision, the field has little understanding of how the different interactions, especially in AR, influence learning. Though prior research documented benefits and challenges of each interaction approach—*e.g.,* tangible interaction promotes student collaboration [14,46] but requires fine motor skills [72], the body of research is small and findings are reported from a specific approach rather than comparing the two approaches. Moreover, much of the research has focused on general user experience (*e.g.,* engagement or cognitive load), not on interaction and its effect on learning. To address this gap, our study compares touchscreen interactions with tangible interactions under the same conditions to support children's mathematizing efforts.

DESIGN PROCESS

To design ARMath, we employed a participatory design process [81] involving teachers, children, and adult designers. Below, we enumerate high-level design goals.

High-Level Design Goals

Informed by prior work [47,77] and our past experience in designing AR learning tools [41–43], we set out to explore five overarching design goals for ARMath.

- ***In situ* visualization of mathematical concepts**. To promote conceptual understanding, ARMath should visualize abstract concepts in objects—*e.g.,* the circular shape of a clock.
- **Use of everyday objects**. We aim to support using everyday objects as math manipulatives and as a means for enacting a specific everyday situation.
- **Contextual math problem.** To promote relevance of learning, math word problems should be contextualized as part of real-life practices.
- **Tangible and virtual interactions.** For problem solving, we aim to offer two interaction options: manipulating physical objects or virtual objects on the touchscreen.
- **Learning goals.** ARMath-based math content and interactions should be aligned with formal elementary mathematics curriculum [65].

Participatory Design (PD) with STEM Teachers

To design ARMath and its learning activities, we conducted two participatory design (PD) sessions with 17 STEM teachers. We collected session video, teacher-created artifacts (*e.g.,* design mockups), and session summaries written by the research team. For analysis, we used thematic coding [10] and peer debrief [84]. Two researchers coded the entire data corpus, followed by peer-debriefing with two other researchers to ensure validity.

Session 1: Design Considerations for AR-based Math

We aimed to uncover design considerations for AR math learning tools. The session included an introduction to AR, critiques of existing AR learning tools, and an all-group discussion. To ground discussions, we demonstrated nine AR learning systems for science and math (*e.g., AR Sandbox* [94], *Photomath* [68]). To solicit feedback, we provided a written template asking about perceived benefits and drawbacks as well as open-ended questions. Teachers suggested the following considerations: (i) provision of adaptive scaffolds for problem solving; (ii) children's opportunity to reflect on their math approaches; (iii) design of mathematically meaningful interactions; and (iv) learner-center approach (*e.g.,* setting individual learning goals, exploring various problems based on their own interests).

Session 2: Co-Designing AR-based Math Content

For Session 2, teachers critiqued ARMath mockups and co-designed new features and learning activities. To scaffold the session, teachers were provided with handouts of math topics for each grade level [65] and ideas cards for facilitating brainstorming. During the critique, teachers were positive about ARMath's potential to turn everyday objects into math manipulatives and promote relevance of learning—e.g., *"ARMath gives opportunity for children to apply mathematics models and see them in action."* A teacher appreciated the potential for learning with large numbers, stating, *"children can practice large numbers without having to get additional materials."* However, teachers shared concerns about technical glitches such as lagging or incorrect object recognition (*e.g., "what if the system says 3 for 4 apples?"*).

In teachers' designs, we identified three emergent themes: (i) providing alternative visualizations; (ii) scaffolding arithmetic operations, and (iii) supporting interactive analysis of shapes. For example, teachers suggested displaying equations for an on-going situation or highlighting geometric primitives (*e.g*., vertices, angles). For arithmetic, they included graphical scaffolds for strategies (*e.g.,* visualizing *equal-number groups* for multiplication) and a monitoring tool that records children's approaches (*e.g., "success or failures on problems, progress tracking")* and reports them back to teachers or parents. For geometry, teachers emphasized inquiry into a real shape (*e.g.,* asking the number of corners in a *STOP* sign), interactive construction (*e.g.,* dragging a book to create a 3D cube), and vocabulary learning.

Participatory Design with Children

Following our PD sessions with teachers, we developed an initial prototype, and conducted two *Cooperative Inquiry (CI)* studies [19] with 8 children (ages 8-12; 5 boys and 3 girls) and 5 adult design partners. In each session, groups of two or three children and adults worked together to test an initial ARMath prototype, elaborate upon each other's ideas, and create designs.

In the first session, we employed a *technology immersion* [34] technique to understand the new approach and brainstorm design ideas. During the test, children recorded their "likes", "dislikes", and "design ideas." Adult partners then synthesized high-level themes and discussed them with all the groups. In the next session, we used the *Bags-of-Stuff* [23] technique in which children use craft supplies (*e.g.,* fabrics, cardboard, markers) to build lo-fi prototypes of their design ideas. After the two sessions, adult partners and researchers synthesized key features from the children's design ideas, which resulted in the following implications.

Extending context in objects. While children liked using everyday objects, more relevant contexts are needed to promote motivation. Children seemed to be engaged with manipulating everyday objects, noting *"like using everyday objects" "would like to use ARMath at home if I can use different kinds of objects."* However, some got bored quickly because there was no context related to *"why do we need to count or add coins."* Children and adult partners suggested presenting virtual situations that involved math operations—*e.g.,* add coins to a bank to buy a toy car.

Repairing AI errors. Because the CV technique for detecting objects and user manipulations sometimes fails, adult partners and researchers agreed on the need for integrating human intervention to identify and correct errors. While children appreciated the AI (*e.g., "like the system know the colors of objects and types of objects"*), they also noticed that the AI could be wrong or slow. A child stated, the *"camera get confused or can't keep up with me moving objects."* These errors led to generating erroneous math problems or rejecting correct answers.

Mobile AR environment. We observed cognitive and behavioral issues related to the mobile AR environment: (i) confusion about a limited view in AR, (ii) less attention on virtual representations, and (iii) distraction by everyday objects. Because the AR camera produces a perspective different from children's eyes, children were confused by gaps between the real world and AR view. For example, when children placed four coins on the table, the camera captured only three and showed incorrect feedback.

ARMATH

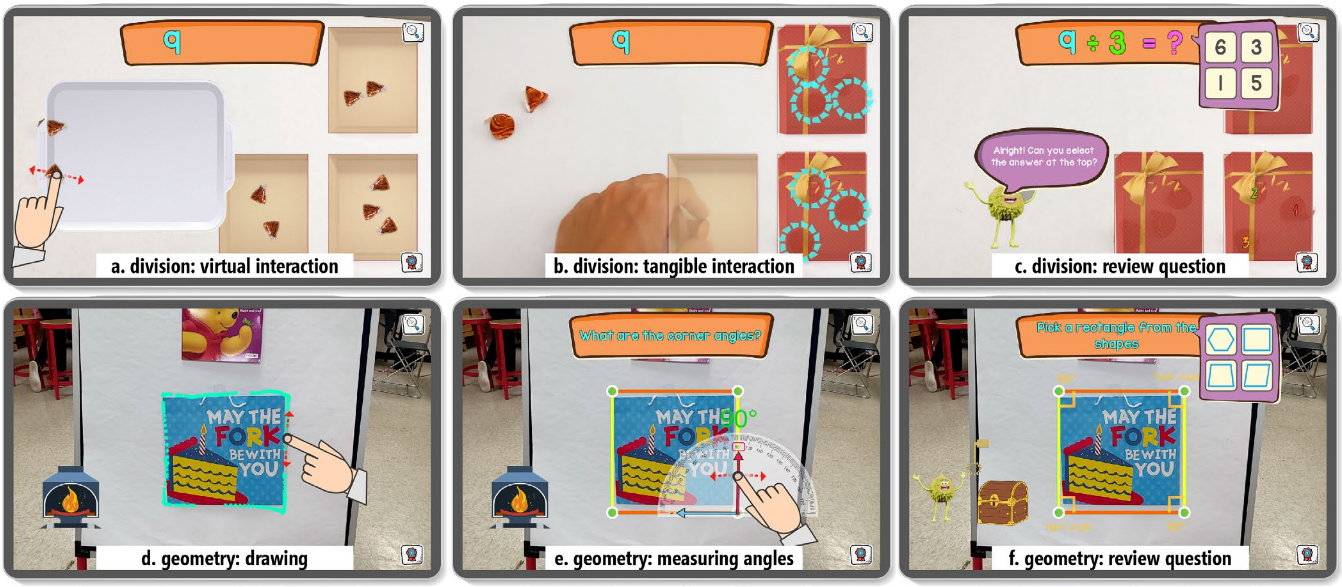
Informed by our PD sessions, we developed the final ARMath system—a mobile AR app—with five application modules for counting, addition, multiplication, division, and geometry (see video). To use ARMath, children find physical objects needed in a virtual situation, putting them in front of the AR camera. Then, children can solve a math problem by manipulating the objects or virtual proxy on the touchscreen. Children can move around with the device to explore objects or sit at a table to interact with found objects.

ARMath Modules

Each module offers a four step user experience: (i) present a virtual and mathematical situation; (ii) find specific everyday objects; (iii) interactively solve a math problem; and (iv) review and solve a formal symbolic problem (Figure 2). To begin, Victor—a friendly virtual 'monster' agent—illustrates a situation that requires math and asks children to find specific everyday objects (*e.g.,* 10 batteries or 8 chocolate candies). Once children place the objects in the AR finder (Figure 3), Victor asks the children to confirm if the objects are recognized correctly and fix any potential errors. Victor then presents a math word problem (*e.g.,* dividing 8 chocolates into 2 groups) and guides children in manipulating the items—either by tangibly moving objects under the AR finder or virtually on the touchscreen (depending on the preconfigured interaction mode). After finishing the operation, children review their work as Victor summarizes the result with numbers, words, and visual cues. Children then solve a formal symbolic problem (*e.g.,* $8 \div 2 = ?$) to ensure they understand the concept before receiving an animated icon as a reward. If children repeat the arithmetic modules, the problems become harder, involving larger numbers. Below, we summarize the five math modules—see the supplementary video for a demonstration.

Counting. As an introductory module, children practice recognizing the number of objects in a group by counting. Victor asks children to find objects and presents a *"how many"* situation. After finding some objects, children count the number of objects by moving (physical or virtual) objects into a virtual tray; the tray displays the on-going count. When all the objects are moved, Victor asks about the number of objects in the tray, highlighting the objects with purple circles—*interactive counters*. The counters enumerate numbers as children tap them.

Addition. Children develop understandings of addition and its connection to counting by counting two sets of objects [64]. Victor asks children to find coins for an ice cream and



a. division: virtual interaction
b. division: tangible interaction
c. division: review question
d. geometry: drawing
e. geometry: measuring angles
f. geometry: review question

Figure 2. In division, after finding 9 chocolates, children divide them equally for three gift boxes. They divide either (a) virtual or (b) the physical chocolates. In the end, (c) children count the number of chocolates in a box (right-bottom) and complete the equation. In geometry, after finding a rectangular bag, children (d) draw the rectangle, identify vertices and sides, and (e) measure corner angles. After reviewing the shape, (f) children identify a rectangle out of four shapes. See our supplementary video.

presents an *"adding to"* situation. A blue rectangle, indicating a set, is overlaid on the objects initially found, and children add a certain number of (physical or virtual) coins to a green rectangle (Figure 5). When finished, Victor asks about the number of coins in the two rectangles, highlighting them with interactive counters.

Multiplication. Children develop understandings of multiplication by representing objects in equal-size groups [64]. Victors asks children to find batteries for Christmas trees and presents a "*successive addition*" situation. Children place a certain number of (physical or virtual) batteries in a box for each tree. When finished, Victor asks about the number of batteries used for all of the trees, highlighting them with interactive counters.

Division. Children understand the meaning of division by distributing the whole number of objects [64]. Victor asks children to find chocolates for gift boxes and presents an "equal sharing" situation. Children place the same number of (physical or virtual) chocolates in each virtual gift box. When finished, Victor asks about the number of chocolates in each box, highlighting them with interactive counters.

Geometry. Children build understandings of rectangular geometry by describing them in an object [64]. Victor asks children to find a rectangular object via an *"investigation"* scenario. Using an image of the found object, children draw a rectangle, identify vertices and sides, and measure corner angles with a virtual protractor. When finished, Victor highlights the components and asks children to identify a rectangle out of four different shapes.

The ARMath System

ARMath system consists of four parts: (i) a *perception engine* that uses CV to recognizes everyday objects, (ii) a *problem generator* that creates storytelling, a math word problem, and a corresponding equation based on the perception, (iii) an *interaction engine* that detects interaction with physical and virtual objects for problem solving and (iv) a *scaffolding engine* that visualizes abstract concepts and helps with math procedures.

Perception engine

To recognize everyday objects and their mathematical attributes, the *perception engine* uses CV and machine learning (ML) including *object detection and tracking* to recognize objects in real-time and *semantic understanding* to draw math information. At any time, children can use the *repairing UI* to correct detection errors.

Object detection and tracking. The first step in the perception process is *object detection* that recognizes all the objects in the camera image, determines the class (*e.g.,* coins, bottles), and estimates the segmented images [35]. We use state-of-the-art object detectors—adding the results from deep learning-based *SSD* [54] and *Mask RCNN* [31] networks to improve the recall—that are robust against scale, perspective, and light. To maintain consistent detection over time, a *multiple object tracker* connects the object instances between video frames, using a common method of iterative prediction and association [6]. To gain robustness against mobility and user action, our tracker suspends the process when movement is detected in gyroscope data or the video stream. The current implementation recognize a set of objects used in the application modules in addition to COCO dataset [50].

Semantic understanding. To draw mathematical information such as set, count, or length, the *semantic understanding* component performs *grouping*, *geometry* *analysis*, and *math inference*. Grouping is a common strategy for whole number concepts and arithmetic operations [11,55]. For grouping, the system detects spatial and color clusters of objects by applying the k-means clustering [30] and GMM classification [67]. For geometry analysis, the system applies contour line analysis [86] and extracts key components such as vertices and sides. The *math inference* component analyzes mathematical attributes of an object using planar tracking [28] and CNN-based regression [73]. For example, it estimates the height of a painting or the water level in a bottle—this is excluded in the modules for low accuracy.

Repairing UI. ARMath involves children in the perception process, allowing for correcting object detection results or geometry shapes. The repairing UI augments objects with visual indicators of *detected-by-camera* and allows children to correct false-positive or false-negative cases by simply tapping them on the screen. Similarly, to rectify errors in geometry analysis, the system offers an optional interface to draw the shape on top of an object (Figure 2d). The system simplifies the hand-drawn shape toward a primitive shape (*e.g.,* straightens a squiggly line).

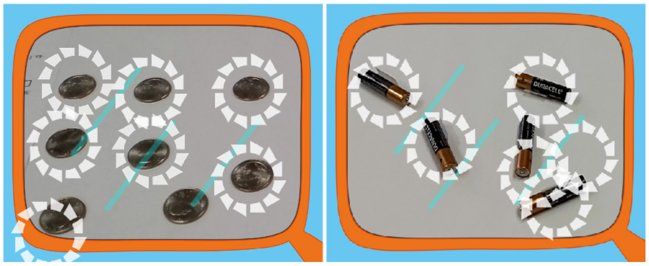


Figure 3. The repairing UI; white circles are overlaid on recognized objects. Children can fix (left) false-negative or (right) false-positive errors by tapping them on the screen.

Problem generator

The problem generator adapts pre-existing graphics and dialogs for storytelling, math word problems, and equations to the current setting of physical objects. All the dialogs are presented both via text and text-to-speech (TTS).

Storytelling. The storytelling engine populates virtual objects, avatars, and dialogs that engage children in a virtual math situation. While storytelling uses static models and animations of virtual objects and avatars, it adapts dialogs to the physical objects involved. The dialogs are implemented as a sequence of speech bubbles that children can interact with to proceed.

Math word problems. During storytelling, the system generates a math word problem. The system adapts a pre-existing problem template to the objects and their math attributes (*e.g.,* count, shapes), and generates a question. For example, in division module (Figure 2), when 8 chocolates are found and a random divisor 2 is selected, the avatar asks, *"We need to distribute the 8 chocolates equality into the two gift boxes. Then, how many chocolates do we have in each box?"* To capture the key information in the problem, an animation highlights both objects in time synchronization with the TTS output.

Equations. In addition to the word problem, the system translates the mathematical situation and presents it abstractly in an equation—*e.g.,* "8 ÷ 2 = ?" This exposes children to symbolic representations, allowing for learning about what equations are composed of and connecting the on-going math operation with the abstract symbol [85].

Interaction engine

ARMath provides two interaction modes for interactive problem solving: *tangible* mode and *touchscreen* mode (Figure 4). These modes are preconfigured and not simultaneously active. In the tangible mode, to perform arithmetic operations, children can place, move, or remove physical objects on the tabletop surface. In the touchscreen mode, for the same operations, children can drag-and-drop multiple virtual objects on the touchscreen. In both modes, the system continuously tracks the user manipulations and translates them into math operations.

Tangible interaction. To support tangible interaction of directly and physically manipulating objects, the system examines the status of individual objects within the AR world and detects the status change. The system examines physical objects' spatial relationships with virtual objects by comparing their positions and areas—*e.g.,* testing if a chocolate is contained in a virtual box. Then, the status result is compared with the previous frames to detect change; the change is regarded as a user manipulation (*e.g.,* adding a chocolate to the box). When a manipulation is detected, the system combines the status results of all objects, translates them into a mathematical representation, and evaluates the representation for providing feedback.

Virtual interaction. To support virtual interaction, the system performs the same process for the tangible interaction, but it considers virtual manipulatives instead. At the beginning, the system creates virtual manipulatives for the existing physical objects. To maintain connection between physical objects and virtual manipulatives, the virtual objects use real-image textures, present on top of the physical objects, and play realistic sounds upon drag-and-drop actions. Moreover, the system duplicates the virtual objects and provides extra manipulatives so that children can operate with large numbers as needed.

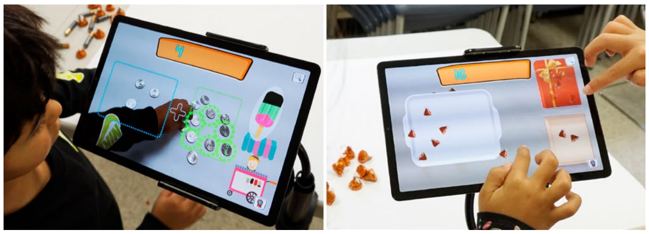


Figure 4. (a) In tangible mode, children use physical coins on the table for addition. (b) In virtual mode, children drag-and-drop virtual chocolates on the touchscreen for division.

Scaffolding Engine

Informed by our PD studies and prior work on scaffolding strategies in learning technology [39,75], ARMath embeds scaffolds including: (i) contextual scaffolds to aid situating math problems in everyday life contexts; (ii) conceptual scaffolds to help understand math concepts; and (iii) procedural scaffolds to guide actions for problem solving.

Contextual Scaffold. The AR imagery, virtual storytelling and the math word problems allow children to think about computations and concepts applicable to a specific life situation. In addition, for children who are more familiar with symbolic equations than story problems [52], the symbolic equations for arithmetic problem are presented.

Conceptual Scaffold. To help children understand math ideas, ARMath augments real objects with graphical representations of abstract concepts such as numbers, sets, and geometry primitives. The graphic is dynamically generated for the manipulatives. For example, in the addition module (Figure 5), the system augments two groups of objects with red and green rectangles respectively so that children can perceive the summation of two distinct sets. As another example (Figure 2f), a rectangle object is augmented with graphics of its vertices, sides, and angles.

Procedural Scaffold. The procedural scaffolds include feedback for user manipulations and virtual tools for numerical counting and measurement. For feedback, the system continuously translates the current status into a mathematical form, and generates feedback based on the evaluation of the form. For example, when children add 2 coins to 5 coin for *"5 + 4 = ?"*, the system prompts, *"add 2 more."* For virtual tools, at the end of arithmetic modules, the system augments (physical or virtual) manipulatives with *interactive counters* that help children count numbers. As children touch a counter, it displays the total count of objects. In the geometry module, children can use a virtual protractor. When children rotate a protractor arm to measure a corner angle, the systems shows the angle value (*e.g., "70°"*) and reads its name (e.g., *"acute angle"*).

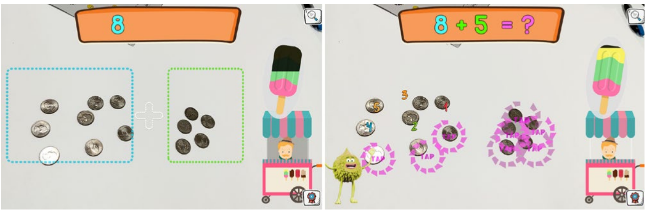


Figure 5. In the addition module, (left) after adding 5 coins (green box) to 8 coins (blue box), children count the total by using the interactive counters (purple circles).

Software Implementation

ARMath is implemented using *TensorFlow* [1] and *OpenCVSharp* [97] for the perception process and *Unity3D/Android* [88] for AR framework. While not limited to a specific device, the application is tested and deployed with the Galaxy Tab S5e.

USER STUDY

To understand how children could use ARMath and to uncover opportunities and challenges therein, we conducted a field deployment at a local children's museum.

Method

Participants were recruited through the museum. We held five identical sessions; 27 children participated (ages 5-8; 14 girls). Children were grouped in age-based pairs though seven children worked alone—for a total of 17 groups. In each session, there were up to four groups of children participants and three adult facilitators. Facilitators helped children use ARMath, provided math knowledge as needed, and conducted a post-play focus group. For one group, a parent stayed with the children for personal reasons.

Each session lasted 80 minutes including an introduction to ARMath and a pre-activity questionnaire (15 min), using ARMath with tangible and virtual interactions (45 min), and a post-activity questionnaire and focus group (20 min). Sessions were conducted at a room with tables. Each table was equipped with a tablet stand. Each group was assigned a table and an ARMath device. Children were allowed to select a math module and move around the room to find and bring everyday objects to the table. Everyday objects recognized by ARMath (*e.g.,* batteries) were provided.

Data and Analysis

We collected questionnaires, session videos, focus group interview recordings, field notes, and system logs. The pre-activity questionnaire examined children's math learning experience (*e.g.,* engagement, use of materials) using child-friendly Likert scales [29] and posed problems designed to elicit their math knowledge. The post-activity questionnaire and focus group included questions about user experience (*e.g.,* fun factor, interaction), self-assessments of learning, and failures in AI. The system logs recorded achievement, interaction, and screenshots.

To analyze the qualitative data, we employed a thematic analysis [10], combined with peer-debriefing [51], where data was iteratively examined and reviewed to identify themes and patterns. Two researchers developed an initial codebook through independent, open coding of data from two different groups. The researchers then worked together in a round of axial coding to clarify, merge, and resolve individual codes, which was followed by a second round of independent coding with the emerging codebook; and another collaborative discussion to resolve disagreements, further clarify details, and finalize the codebook. Finally, two researchers split the field study data to synthesize and triangulate findings across all data sources.

Findings

We present findings related to user engagement, scaffolds, interaction modes, experiences with failures in AI, learning potential, and challenges. For Likert questions (scale: 1-5, 5 is best), we report means (*M*) and standard deviations (*SD*).

Engagement. The "engagement" theme emerged from our observations of children using ARMath and what attributes supported their engagement. On the post-activity questionnaire, most children indicated having fun with ARMath; 19 out of 27 children gave 4 or 5 (*M*=4.1; *SD*=1.3) to the question *"Using ARMath is fun."* In the follow-up interview, children liked using everyday objects (*e.g., "It was really fun because I'm using real objects"*), life-relevant actions (*e.g., "I liked division because I like dividing things"*), and visualizations (*e.g., "I liked the numbers on the screen"*). However, four children shared negative reactions; three of whom were on the younger end of our age range: 5-6 years old. For example, one child (age 5) commented, "*I don't like shapes because I don't understand it*." Further work is needed to identify what additional scaffolds might help younger learners understand solve these more complex problems.

We observed that several children reinforced concepts by repeating modules. Children often repeated the same module back-to-back, trying new objects or challenging themselves with a harder problem (*e.g.,* more objects to count or divide). For example, a group did the geometry module three times in a row, collecting a variety of rectangular objects (*e.g.*, painting, worksheet, and envelope; Figure 6). In another group, after finishing a multiplication module, a child was excited to tackle a harder problem, saying, *"Hey, we can do it again, we can do it more, I guess it goes harder."*

Figure 6. With the geometry module, a group explored three different rectangular objects.

Our video analysis revealed that our storytelling approach engaged children emotionally. They expressed surprise, responded quickly to system prompts, and were motivated to perform math tasks. Most children appeared immersed in the virtual situation and worked hard to help Victor address his math problems. For example, when Victor asks for more coins to buy ice cream, all the children were quick to add some coins. Having successfully completed an addition module, many children chose to repeat their accomplishment, expressing surprise that Victor would then demand a larger number of coins: *"Oh my God! Eleven! We need eleven coins! Really?,"* Another child emphasized the narrative context for the multiplication module, stating *"I liked multiplication because I needed to take batteries to turn on the trees."*

Scaffolds. We examined how children used the scaffolds present in ARMath and what scaffolds facilitators supplied *in-situ*. Our video analysis showed that children used *interactive counters* to help them find solutions and that equations triggered conversations about formal symbolic math. For example, when the formula "2 x 4 = ?" is introduced, one group initially answered "6." After realizing this was incorrect, one child used the interactive counter to count along, "*one, two, three…eight!*" before correctly selecting "8." Others used the counter to verify their answers, while two groups that had correctly calculated their answers from equations also seemed to check their solutions by slowly counting the objects aloud.

Our video analysis indicated that ARMath's approach of showing virtual representations alongside concrete physical representations, overlaid by symbolic notation (e.g., "÷" operator) prompted math discourse and supported children's sense-making efforts. For example, when the equation "6 ÷ 2 = ?" was shown, an older brother made the connections for his sister via the interface, pointing out, "*Do you know what 6 divided by 2 is? … So, 6 divided by 2 is three because putting three two times equals six.*" Similarly, another child asked about the multiplication operator, "*What is this X?*" after completing two rounds of the multiplication module; a facilitator explained.

The interactive protractor seemed to be the most engaging feature of the geometry module. We observed that 11 out of 17 groups played with the protractor needle to explore different angles, often reading aloud with the ARMath verbal scaffold. For example, after trying 5 different angles with the protractor, one child observed, "*When it goes over this (90 degree), it is hmm Obtuse angle!"* and *"This is acute. Is it because it is less than the right angle?"*

We observed that facilitators offered three types of scaffolds: (i) providing domain knowledge (*e.g.,* geometry vocabulary); (ii) explaining AI limitations with metaphors (*e.g., "The computer's brain is tired", "It cannot see stacked coins"*), and (iii) directing children's attention to the virtual agent (e.g., *"What does the puffy guy say?"*)

Tangible and Virtual interactions. Our results show little difference in preference or children's natural approach. In the post-activity questionnaire, children showed equally high preference for the two interaction modes; they gave a mean rating of 4.2 (*SD*=1.3) for the tangible and 4.4 (*SD*=1.1) for the virtual. One child noted that virtual manipulation afforded the same interaction as the tangible one, "*I liked moving (virtual) objects on the screen because we can move them anywhere like on the table.*" Also, we did not observed tendency in children's natural approaches. Because our participants had little experience with tablets or AR, we assumed that children preferred physical manipulation over virtual. However, we did not see significant differences between or within groups.

We observed notable differences in the pace of arithmetical operations and collaboration. In our video analysis, children took a rapid and single-step approach in tangible mode, whilst they took a slow and multi-steps approach in virtual mode. For example, when prompted to move a group of 4 batteries, a child quickly placed a handful of 7 batteries and promptly adjusted upon the system's feedback (*e.g.,* "*too many").* Conversely, despite the ability to move multiple virtual objects concurrently, the child carefully moved batteries one by one, counting aloud until he got the right number. Interestingly, collaborative operations occurred more frequently in virtual mode. For example, one group split division tasks, saying "*Now you take two on that, and now I take two on the other.*" Then, they took turns dragging-and-dropping the virtual chocolate in the boxes. In the later tangible division, only one child distributed chocolates quickly but in a less organized way.

Failures in AI. We analyzed how children understood and reacted to object recognition errors and their thoughts about the "imperfect" AI. While most children experienced several occurrences of recognition errors, they also seemed to understand ARMath's AI constraints. Children then helped the system recognize objects by placing objects more appropriately and waited patiently rather than expressing frustration. For example, once facilitators explained ways to help Victor (the virtual agent), most children tried to spread objects so that the system could distinguish adjacent objects. Children even gave Victor up to 20 seconds to recognize objects—*e.g.,* a group screamed with joy after waiting 5 seconds. However, one group that was not explicitly told the AI "sometimes makes mistakes seeing" struggled to manipulate objects (*e.g.,* moving the tablet vs. object; holding an object too close to the camera).

With the repairing UI, most children quickly fixed the false-negative detection errors, but they showed negative reactions to false-positive ones. At the beginning, children were told *"you can help Victor because he does not see very well."* During the study, they immediately fixed unrecognized objects and seemed happy with the systems' reaction—e.g., *"Hey look, now he sees it."* Surprisingly, few children ignored the errors. However, when Victor indicates false existence of objects, children expressed negative reactions, thinking Victor was lying (*e.g.,* a child complained, *"he circled (recognized) when it was not there").*

In the focus group, we asked what children thought of helping correct Victor's errors. While two groups shared negative experiences (*e.g., "He was wrong often. I found it annoying when I had to help him"*), eight groups liked to help (*e.g., "Everyone makes mistakes and learns from the mistakes. People like helping"*). Moreover, three groups indicated that they learned from repairing errors. One child said, "*He was a little confused about the math. I think I helped him and I learned some when I helped him.*"

Learning Potential. Our exploratory evaluation consisted of a single 80 minute session with each group, so achieving or measuring learning outcomes was not a primary goal. However, our analysis indicates ways that ARMath could contribute to learning. In the post-activity questionnaire, 22 of 25 children agreed *"ARMath helped learn math"* (*M*=4.2; *SD*=1.0). Specifically, children indicated that ARMath reinforced arithmetic operations (*e.g., "I think I learned a bit more about division*") and symbolic notation (*e.g., "I learned numbers", "The symbol. I forgot the name of the symbol"*). With ARMath, children wanted to learn more operations (*e.g., "minus, not just plus"*), measurement (*e.g., "length"*), and other shapes (*e.g., "Hexagon"*).

Our video analysis highlighted a potential to promote children's motivation and confidence. Children's comfort and familiarity with everyday objects motivated play with larger numbers or different shapes. For example, one child explored double-digit addition because she *"just wanted to have a lot of coins."* ARMath also seemed to encourage children's confidence by allowing them to solve otherwise difficult math problems on their own. As one child explained, *"ARMath makes me learn better. I struggled with division at home. I learned about division."* Another child boasted, *"This is my second problem. Dad see, look, I did these two (counting and addition)."*

Challenges. We observed three primary challenges: (i) issues with hand-eye coordination [12,72]; (ii) discrepancies between children's conception of a shape and how it looked in AR view; and (3) insufficient conceptual scaffolds. We observed that most children experienced difficulties with hand-eye coordination, as the mobile AR environment makes coordinating physical movements through an AR screen more difficult. In particular, children struggled to place physical objects at the right place on-screen. In response, some children devised a collaborative solution: in three groups, children split tasks so one child manipulated physical objects while the other monitored the AR screen. One child directed, "*I will keep an eye on the screen, I will tell you what batteries you move*" (Figure 7).

The geometry module's system logs showed that children struggled with perspective distortion. The AI performs geometry analysis best when an object is as close to a true rectangle shape as possible. Consequently, both system and facilitator prompt children to take pictures in this way. However, children often ignored the instructions or failed to notice the AI made a distortion error (Figure 6 right). Children paid little attention to the object's on-screen presentation; rather, they stuck to their conception that the physical object was a rectangle, despite the AI errors.

Figure 7. (Left) a child struggled with adjusting physical interaction to the AR view. (Right) two children split tasks between physical and virtual surfaces.

DISCUSSION AND CONCLUSION

In this paper, we introduce ARMath to support mathematization experiences in everyday life. Leveraging CV and AR, ARMath recognizes physical objects, enacts a mathematical situation, and supports interactive problem solving or geometry analysis. Through participatory design with teachers and children, we elicited design ideas useful for ARMath as well as general AR-based STEM tools. Our user study allowed us to understand how children engage with everyday objects for learning, their interaction patterns in tangible and virtual surfaces, and uncovered new opportunities of child-AI interaction for learning. While ARMath demonstrates the potential of AR for everyday math, more work is needed to address usability issues, design effective child-AI interaction, and evaluate learning.

AR-interactive storytelling. Our findings revealed an opportunity for AR storytelling to engage children in mathematization. These findings extend the benefits of AR storytelling—previously limited to literacy education [7], edutainment [38], and journalism [66]— to math learning. ARMath's interactive story enabled children to actively participate in meaningful math tasks using everyday objects in familiar contexts. This affirms Billinghurst *et al.*'s design requirement that "*interaction beyond navigation*" is essential for compelling AR experiences [8].

Bridging concrete and abstract math. Our findings demonstrate an opportunity of AR visualization to bridge the gap between hands-on math activities and formal symbolic math. Translating mathematical situations into abstract representations is critical in elementary school mathematics [15]. To our knowledge, however, little research has shown how hands-on learning with manipulatives helps children make conceptual connections between abstract and symbolic representations [62]. Our findings suggests that showing abstract equations in AR can trigger children's interest or reinforce explicit connections between the symbolic and concrete—*e.g.,* children questioned the symbols or explained the equations to peers.

Opportunistic use of everyday objects. Prior work in AR UIs explored how everyday objects enrich haptic experience [33] or controller interfaces [32]; however, little work has focused on how they can be used for learning. We have only begun to explore the opportunity of everyday objects as manipulatives for children's math learning. Our findings affirm Liu *et al.'s* suggestion that using real-world manipulatives can be generally helpful for learning [53], as well as Mbogho *et al.*'s claim indicating that students can be engaged with actual physical objects [60]. Our work extends this knowledge by showing how everyday objects can be engaging manipulatives and prompt playful, story-based mathematizing in familiar, meaningful contexts.

Child-AI Interaction. Child-AI interaction can be characterized by a high probability of failures (*e.g.,* conversation breakdowns with Alexa [4]) and children's conception of machines as "like a person" [56]. Our work extends the knowledge by examining children's reactions, attitudes, and efforts to repair system errors in learning contexts. We found that, with facilitators' help, children could understand AI behaviors and adapted their manipulations to system recognition limitations. These findings support Beneteau *et al.*'s claim that youth can understand machine learning (ML) behaviors [4], with adult mediation, as suggested by Cheng *et al* [16,96]. At times children still reacted negatively to the AI's behavior of the false-positive errors (*e.g.,* similar to *creepy* deception [96]), which suggests the need for higher precision and recall [70] in CV and ML techniques for learning contexts.

Furthermore, our findings regarding children's efforts to repair AI errors suggest a new opportunity for learning. Our observations of children's persistent engagement affirm Cheng *et al.*'s [16] finding that repairing mechanism is essential for children's persistent use of conversational AI and extend it to vision-based learning applications. In our study, when children took steps to repair AI errors, they had an opportunity to evaluate the AI's mathematical misunderstandings and learn by correcting them. As a result, two children explicitly mentioned 'correcting Victor' as an avenue for learning (*e.g., "I learned some when I helped him").* Future work may explore designs or learning activities that can leverage this child-AI interaction and study potential cognitive processes involved.

Virtual vs. Tangible manipulatives. Our work contributes to research attempting to compare children's use of tangible and virtual manipulatives in math education [9,58,63]. Unlike prior work, however, our AR approach afforded the opportunities to compare the two modalities in the same mixed-reality environment. While children showed little difference in their preferences, our findings indicate that the touchscreen interaction promotes collaboration and reflection by slowing down children's actions. We attribute these results to the touchscreen's physical constraints (in terms of space and action), giving credence to Manches *et al.*'s [58] claim that manipulative characteristics of interfaces can influence children's numerical strategies. Our work extends this knowledge by demonstrating how slower-paced, space-constrained virtual interfaces can encourage collaborative math learning.

Limitations and future work. While our work demonstrates the potential of AR and everyday objects to promote mathematization, our study has limitations related to usability, the repairing UI, and parent/teacher facilitation. Our mobile AR approach highlighted issues related to hand-eye coordination, discrepancies between children's perception and AR view, and stabilizing the device, which may limit practical use cases. More immersive devices such as *HoloLens* or *AR glasses* may address these limitations. In addition, more effective repairing schemes need to be designed to integrate AI capabilities in learning tools. Lastly, future work may explore when and how to involve parents or teachers in children's mathematization efforts.

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